

REMARKS ON GLOBAL GEODESIC PROPERTIES OF GÖDEL TYPE SPACETIMES

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ABSTRACT. The aim of this paper is to review and complete the study of geodesics on Gödel type spacetimes initiated in [8] and improved in [2]. In particular, we prove some new results on geodesic connectedness and geodesic completeness for these spacetimes.

1. INTRODUCTION

Classical critical points theorems and standard Morse theory are directly applicable to functionals bounded from below which satisfy compactness assumptions, such as the Palais–Smale condition (see Section 2), and whose critical points have finite Morse index. Unluckily, these tools cannot be applied to many interesting problems involving functionals that are strongly indefinite. For example, geodesics joining two points z_p, z_q on an indefinite semi–Riemannian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ are the critical points of the strongly indefinite C^1 action functional

$$f(z) = \int_0^1 \langle \dot{z}, \dot{z} \rangle_L ds \quad (1.1)$$

defined on the Hilbert manifold Ω of all the H^1 –curves joining z_p to z_q in \mathcal{M} (for more details, see Section 2). Anyway, starting from the seminal paper [4], in some particular settings, and according to the properties of the manifold \mathcal{M} and its indefinite metric $\langle \cdot, \cdot \rangle_L$, the functional f in (1.1) has been widely studied by using variational methods, also obtaining sometimes optimal results at least in the Lorentzian case (we refer to the updated survey paper [10] and references therein). A typical situation occurs when the Lorentzian metric tensor $\langle \cdot, \cdot \rangle_L$ presents symmetries (i.e., Killing vector fields): one gets rid of the negative contributions in the directions of the Killing fields and, by means of some variational principles, it is possible to handle with simpler functionals, which essentially depend only on a Riemannian metric, so that they are bounded from below and satisfy the Palais–Smale condition under reasonable assumptions. This is the case of standard stationary and Gödel type spacetimes.

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Definition 1.1. A Lorentzian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ is a *standard stationary space-time* if there exist a smooth finite-dimensional Riemannian manifold $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$, a vector field δ and a positive smooth function β on \mathcal{M}_0 such that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and the Lorentz metric (under natural identifications) is

$$\langle \cdot, \cdot \rangle_L = \langle \cdot, \cdot \rangle_R + 2\langle \delta(x), \cdot \rangle_R dt - \beta(x) dt^2. \quad (1.2)$$

When the cross term vanishes ($\delta \equiv 0$) the spacetime is called *standard static*. This is a warped product $\mathcal{M}_0 \times_{\sqrt{\beta}} \mathbb{R}$ with Riemannian base and negative definite fiber.

Recall that every stationary spacetime (i.e., a spacetime admitting a timelike Killing vector field K) is locally a standard stationary one with $K = \partial_t$.

On the other hand, Gödel type spacetimes are Lorentzian manifolds admitting a pair of commuting Killing vector fields which span a two dimensional distribution where the metric has index 1 (the causal characters of the Killing vectors could change on the manifold, see [11, Example 5.1]). More precisely, we use the following definition (according to [8]):

Definition 1.2. A Lorentzian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ is a *Gödel type spacetime*, briefly (*GTS*), if a smooth finite-dimensional Riemannian manifold $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$ exists such that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ and the metric $\langle \cdot, \cdot \rangle_L$ is described as

$$\langle \cdot, \cdot \rangle_L = \langle \cdot, \cdot \rangle_R + A(x)dy^2 + 2B(x)dydt - C(x)dt^2, \quad (1.3)$$

where $x \in \mathcal{M}_0$, the variables (y, t) are the natural coordinates of \mathbb{R}^2 and A, B, C are C^1 scalar fields on \mathcal{M}_0 satisfying

$$H(x) = B^2(x) + A(x)C(x) > 0 \quad \text{for all } x \in \mathcal{M}_0. \quad (1.4)$$

Let us observe that condition (1.4) implies that metric (1.3) is Lorentzian. It is also interesting to point out that (*GTS*) are not necessarily time-orientable (e.g., cf. [11, Remark 1.2]).

In [18] Gödel gives an exact solution of Einstein's field equations with homogeneous perfect fluid distribution, the so-called classical Gödel Universe. This spacetime, described in Example 1.3(e_1) below (see also [14, 17] where its geodesic equations are explicitly integrated), has a five dimensional group of isometries, is geodesically complete and admits closed causal curves (e.g., cf. [15]). In [23] Raychaudhuri and Thakurta start the study of homogeneity properties of (*GTS*) investigating homogeneity conditions of a class of cylindrically symmetric metrics; later on, in [24] Rebouças and Tiomno introduce a definition for Gödel metrics in four dimensions and study their homogeneity conditions (see also [6, 13]).

Example 1.3. The class of (*GTS*) depicted in Definition 1.2 is wide; indeed this definition covers very different kinds of spacetimes, including some physically relevant examples.

(e_1) The Gödel Universe (cf. [18]) is a (*GTS*) with

$$\mathcal{M}_0 = \mathbb{R}^2, \quad \langle \cdot, \cdot \rangle_R = dx_1^2 + dx_2^2$$

and with coefficients in (1.3) given by

$$A(x) = -e^{2\sqrt{2}\omega x_1}/2, \quad B(x) = -e^{\sqrt{2}\omega x_1}, \quad C(x) \equiv 1$$

($\omega > 0$ represents the magnitude of the vorticity of the flow). In [8], by a direct integration of the geodesic equations, it is constructed a geodesic joining each couple of points in \mathcal{M} .

- (e₂) The Gödel–Synge spacetimes (cf. [26]) are (*GTS*) with $\mathcal{M}_0 = \mathbb{R}^2$ and

$$\langle \cdot, \cdot \rangle_L = dx_1^2 + dx_2^2 - g(x_1)dy^2 - 2h(x_1)dydt - dt^2,$$

where g, h are smooth functions of x_1 with $g > 0$. If $2g = h^2$ and $h = e^{x_1}$, this metric reduces to the Gödel classical one.

- (e₃) Some Kerr–Schild spacetimes (e.g., cf. [16]) are (*GTS*) with again $\mathcal{M}_0 = \mathbb{R}^2$ and

$$\langle \cdot, \cdot \rangle_L = dx_1^2 + dx_2^2 + dy^2 - dt^2 + V(x_1, x_2)(dy + dt)^2,$$

where V is an arbitrary function on \mathbb{R}^2 . In this case, the coefficients in (1.3) are

$$A(x) = 1 + V(x), \quad B(x) = V(x), \quad C(x) = 1 - V(x),$$

and thus, $H(x) \equiv 1$.

- (e₄) Some standard stationary spacetimes are (*GTS*) with $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$, being $(\mathcal{M}_0 \times \mathbb{R}, \langle \cdot, \cdot \rangle_R + dy^2)$ the Riemannian part and

$$\langle \cdot, \cdot \rangle_L = \langle \cdot, \cdot \rangle_R + dy^2 + 2\delta(x)dydt - \beta(x)dt^2$$

the stationary metric with $\delta(x, y) \equiv \delta(x) \in \mathbb{R}$ and $\beta(x, y) \equiv \beta(x) > 0$ in $\mathcal{M}_0 \times \mathbb{R}$. Clearly, they are (*GTS*) with $A(x) \equiv 1$, $B(x) = \delta(x)$ and $C(x) = \beta(x)$.

Vice versa, some (*GTS*) are standard stationary spacetimes with the product $A(x)C(x) > 0$ on \mathcal{M}_0 , being standard static if, in addition, $B \equiv 0$. For example, if $A(x) > 0$ on \mathcal{M}_0 , the spatial part of the stationary spacetime corresponds to $\mathcal{M}_0 \times \mathbb{R}$ equipped with the Riemannian metric $\langle \cdot, \cdot \rangle_R + A(x)dy^2$ (which is complete if so is $\langle \cdot, \cdot \rangle_R$), the vector field becomes $\delta(x, y) = (0, B(x)) \in T\mathcal{M}_0 \times \mathbb{R}$ and the scalar field is $\beta(x, y) = C(x) > 0$ for each $(x, y) \in \mathcal{M}_0 \times \mathbb{R}$.

- (e₅) Some examples of general plane fronted waves are also (*GTS*). More precisely, a *general plane fronted wave* is a Lorentzian manifold $\mathcal{M}_0 \times \mathbb{R}^2$ equipped with the metric

$$\langle \cdot, \cdot \rangle_L = \langle \cdot, \cdot \rangle_R + 2dydt + H_0(x, t)dt^2,$$

where $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$ is a Riemannian manifold, (y, t) are the natural coordinates of \mathbb{R}^2 and the smooth scalar field H_0 on $\mathcal{M}_0 \times \mathbb{R}$ satisfies $H_0 \not\equiv 0$. Clearly, when $H_0(x, t)$ is autonomous (i.e., it does not depend on t), this spacetime is a (*GTS*).

Results on geodesic completeness and connectedness for these spacetimes can be found in [7].

The importance of the spacetimes above justifies the study of global properties such as geodesic connectedness and geodesic completeness. However, one cannot expect to prove general results: indeed the global properties depend on the behaviour of the coefficients (see respectively Theorems 3.1 and 4.1 and related comments). Also the study of global hyperbolicity of (*GTS*) is interesting but, except in the case of classical Gödel Universe, this is still an open problem even if we are lead to think that at least the chronology condition should be verified. This idea comes out as in most of the previous references, when global properties can be studied by variational methods, then the stated assumptions imply also global hyperbolicity (cf. [10] and references therein). In fact, for example in the particular case of

standard stationary spacetimes, we know sufficient conditions, involving the coefficients of the metric, which ensure global hyperbolicity (cf. [25, Corollary 3.4]), while recently a characterization has been proven (see [12, Theorem 4.3]).

The paper is organized as follows. In Section 2 we recall some variational principles for geodesics on static spacetimes and (GTS) . In Section 3 we present a new result on geodesic connectedness, and compare it with the previous ones in [2], showing its accuracy by examples. In Section 4 we deal with geodesic completeness, and state a sufficient condition in order to obtain it. Finally, in the Appendix we fix some widely known notations about the variational set up.

2. THE VARIATIONAL PRINCIPLE

According to notations and statements contained in the Appendix, there is a correspondence between geodesics joining two given points z_p, z_q on a semi-Riemannian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ and critical points of the action functional f in (1.1) on the Hilbert manifold $\Omega^1(z_p, z_q)$. As already remarked, if $\langle \cdot, \cdot \rangle_L$ is not Riemannian then f is strongly indefinite, but, in some Lorentzian manifolds, this difficulty can be overcome by introducing a new suitable functional.

The kernel of our approach is a variational principle stated in [5, Theorem 2.1] for static Lorentzian manifolds $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$, with $\langle \cdot, \cdot \rangle_L$ as in (1.2) and $\delta \equiv 0$. It is based on the fact that $\langle \partial_t, \dot{z} \rangle_L$ is constant along each geodesic z , because of the Killing character of ∂_t . Namely, $z_p = (x_p, t_p)$, $z_q = (x_q, t_q) \in \mathcal{M}$ are connected by a geodesic $\bar{z} = (\bar{x}, \bar{t})$, which is a critical point of functional f in (1.1) on $\Omega^1(z_p, z_q) = \Omega^1(x_p, x_q) \times W(t_p, t_q)$, if and only if \bar{x} is a critical point of the functional

$$J(x) = \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle_R ds - \frac{\Delta_t^2}{2} \left(\int_0^1 \frac{1}{\beta(x)} ds \right)^{-1} \quad \text{on } \Omega^1(x_p, x_q), \quad (2.1)$$

with $\Delta_t := t_p - t_q$ (for the stationary case we refer to [10, Section 4.2] and references therein).

Next, let us consider the more general setting of (GTS) with $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ and $\langle \cdot, \cdot \rangle_L$ as in Definition 1.2. For each $x \in H^1(I, \mathcal{M}_0)$ let us define

$$a(x) = \int_0^1 \frac{A(x)}{H(x)} ds, \quad b(x) = \int_0^1 \frac{B(x)}{H(x)} ds, \quad c(x) = \int_0^1 \frac{C(x)}{H(x)} ds, \quad (2.2)$$

$$\mathcal{L}(x) = b^2(x) + a(x)c(x). \quad (2.3)$$

As every (GTS) admits two Killing vector fields ∂_y, ∂_t (not necessarily timelike), an extension of the previous variational principle can be stated (cf. [8, Proposition 2.2]). In this setting, fixing $z_p = (x_p, y_p, t_p)$, $z_q = (x_q, y_q, t_q) \in \mathcal{M}$, with $x_p, x_q \in \mathcal{M}_0$ and $(y_p, t_p), (y_q, t_q) \in \mathbb{R}^2$, we have that $\bar{z} : I \rightarrow \mathcal{M}$ is a geodesic joining z_p to z_q in \mathcal{M} if and only if it is a critical point of the action functional (1.1), with $\langle \cdot, \cdot \rangle_L$ as in (1.3), defined on the manifold $\Omega^1(z_p, z_q) = \Omega^1(x_p, x_q) \times W(y_p, y_q) \times W(t_p, t_q)$.

Let $x \in \Omega^1(x_p, x_q)$ be such that $\mathcal{L}(x) \neq 0$ (cf. (2.3)). For all $s \in I$ we define

$$\begin{aligned}\phi_y(x)(s) &:= y_p + \frac{\Delta_y b(x) - \Delta_t c(x)}{\mathcal{L}(x)} \int_0^s \frac{B(x)}{\mathcal{H}(x)} d\sigma \\ &\quad + \frac{\Delta_y a(x) + \Delta_t b(x)}{\mathcal{L}(x)} \int_0^s \frac{C(x)}{\mathcal{H}(x)} d\sigma, \\ \phi_t(x)(s) &:= t_p - \frac{\Delta_y b(x) - \Delta_t c(x)}{\mathcal{L}(x)} \int_0^s \frac{A(x)}{\mathcal{H}(x)} d\sigma \\ &\quad + \frac{\Delta_y a(x) + \Delta_t b(x)}{\mathcal{L}(x)} \int_0^s \frac{B(x)}{\mathcal{H}(x)} d\sigma\end{aligned}$$

with $\Delta_y := y_q - y_p$ and $\Delta_t := t_q - t_p$. Standard arguments imply that the functions ϕ_y and ϕ_t , which go from $\Omega^1(x_p, x_q)$ to $W(y_p, y_q)$ and $W(t_p, t_q)$, resp., are C^1 .

Then, the following proposition holds (see [8, Proposition 2.2]).

Proposition 2.1. *Let $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ be a (GTS) and $x_p, x_q \in \mathcal{M}_0$ be such that $|\mathcal{L}(x)| > 0$ for all $x \in \Omega^1(x_p, x_q)$. Then, the following statements are equivalent:*

- (i) $\bar{z} \in \Omega^1(z_p, z_q)$ is a critical point of the action functional f in (1.1);
- (ii) setting $\bar{z} = (\bar{x}, \bar{y}, \bar{t})$, the curve $\bar{x} \in \Omega^1(x_p, x_q)$ is a critical point of the C^1 functional

$$\mathcal{J}(x) = \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle_R ds + \frac{\Delta_y^2 a(x) + 2\Delta_y \Delta_t b(x) - \Delta_t^2 c(x)}{2\mathcal{L}(x)} \quad \text{on } \Omega^1(x_p, x_q) \quad (2.4)$$

(see (2.2)–(2.3)), while $\bar{y} = \phi_y(\bar{x})$, $\bar{t} = \phi_t(\bar{x})$, with ϕ_y, ϕ_t as above.

Furthermore,

$$\mathcal{J}(x) = f(x, \phi_y(x), \phi_t(x)) \quad \text{for all } x \in \Omega^1(x_p, x_q).$$

Thus, the geodesic connectedness problem in the standard static and (GTS) cases reduces to give conditions on the functionals J in (2.1) and \mathcal{J} in (2.4), resp., which allows us to apply the classical critical point theorem below (see [22, Theorem 2.7]).

Theorem 2.2. *Assume that Ω is a complete Riemannian manifold and F is a C^1 functional on Ω which satisfies the Palais–Smale condition, i.e. any sequence $(x_k)_k \subset \Omega$ such that*

$$(F(x_k))_k \text{ is bounded and } \lim_{k \rightarrow +\infty} F'(x_k) = 0,$$

converges in Ω , up to subsequences. Then, if F is bounded from below, it attains its infimum.

Remark 2.3. In order to obtain a multiplicity result on geodesics joining two fixed points in standard static spacetimes or (GTS), the Ljusternik–Schnirelman theory can be applied to J in (2.1) or \mathcal{J} in (2.4) whenever the Riemannian part has a “rich topology” (for the static case see [3] and references therein, for (GTS) see [2, 8, 9]).

In order to avoid technicalities, hereafter we assume that \mathcal{M}_0 is complete, so that $\Omega^1(x_p, x_q)$ is also complete for each $x_p, x_q \in \mathcal{M}_0$. Then, let us recall the following result (cf. [3, Proposition 4.3] and [2, Lemma 5.3]).

Lemma 2.4. *Let x_p, x_q be two points of a given complete Riemannian manifold $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$. Then, the following statements hold:*

- (a) if $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ is a static Lorentzian manifold and J in (2.1) is coercive on $\Omega^1(x_p, x_q)$, then J satisfies the Palais–Smale condition on $\Omega^1(x_p, x_q)$;
- (b) if $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ is a (GTS), \mathcal{J} in (2.4) is coercive on $\Omega^1(x_p, x_q)$, and there exists $\nu > 0$ such that

$$|\mathcal{L}(x)| \geq \nu \quad \text{for all } x \in \Omega^1(x_p, x_q),$$

then \mathcal{J} satisfies the Palais–Smale condition on $\Omega^1(x_p, x_q)$.

Summing up, geodesic connectedness of the mentioned spacetimes is guaranteed by conditions implying the coercivity and lower boundedness of the “Riemannian” functional associated to the problem.

For instance, in the case of J in (2.1), these conditions correspond to restrictions on the growth of the (positive) metric coefficient β in (1.2): β bounded in the pioneer paper [5] or, more in general, β subquadratic or growing at most quadratically with respect to the distance $d(\cdot, \cdot)$ induced on \mathcal{M}_0 by its Riemannian metric $\langle \cdot, \cdot \rangle_R$, i.e. existence of $\lambda \geq 0$, $k \in \mathbb{R}$ and a point $x_0 \in \mathcal{M}_0$ such that

$$0 < \beta(x) \leq \lambda d^2(x, x_0) + k \quad \text{for all } x \in \mathcal{M}_0, \quad (2.5)$$

(cf. [3, Theorem 1.1] and references therein). Remarkably, this second growth condition on β is optimal, as showed in [3, Section 7] by constructing a family of geodesically disconnected static spacetimes with superquadratic, but arbitrarily close to quadratic, coefficients β .

3. GEODESIC CONNECTEDNESS IN (GTS)

At a first glance the problem in (GTS) can be handled in the same manner as in the static case. However, we cannot expect optimality by applying this variational approach. In fact, the classical Gödel Universe cannot be studied by our tools, due to the lack of the assumption $\mathcal{L}(x) \neq 0$ on $\Omega^1(x_p, x_q)$ for each couple of points $x_p, x_q \in \mathbb{R}^2$ (cf. Example 1.3(e₁)). In this section we state and prove a new theorem on geodesic connectedness for (GTS) (in addition to the previous ones in [2, 8]), which, even if is not optimal, is accurate in the sense described below (see Corollary 3.2 and Example 3.3).

Theorem 3.1. *Let $(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2, \langle \cdot, \cdot \rangle_L)$ be a Gödel type spacetime such that:*

- (h₁) $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$ is a complete Riemannian manifold;
 - (h₂) there exists $\nu > 0$ such that $\mathcal{L}(x) \geq \nu > 0$ for all $x \in H^1(I, \mathcal{M}_0)$;
 - (h₃) $m(x) \geq h(x) > 0$ for all $x \in H^1(I, \mathcal{M}_0)$, with $m(x) := \max\{a(x), -c(x)\}$
- and

$$h(x) := \int_0^1 \frac{ds}{\lambda d^2(x(s), x_0) + k} \quad \text{for some } \lambda \geq 0, k \in \mathbb{R} \text{ and } x_0 \in \mathcal{M}_0.$$

Then, $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ is geodesically connected.

Proof. Let us take any $z_p = (x_p, y_p, t_p)$, $z_q = (x_q, y_q, t_q) \in \mathcal{M}$, with $x_p, x_q \in \mathcal{M}_0$ and $(y_p, t_p), (y_q, t_q) \in \mathbb{R}^2$. From hypothesis (h₂) (in particular $\mathcal{L}(x) \neq 0$), Proposition 2.1 can be applied, and so the existence of geodesics joining z_p to z_q reduces to find some critical points of \mathcal{J} in (2.4) on $\Omega^1(x_p, x_q)$. Following the arguments developed in [2, Section 5], we have that \mathcal{J} can be written as follows:

$$\mathcal{J}(x) = \frac{1}{2} \|\dot{x}\|^2 - \frac{1}{2} \frac{\Delta_+^2(x)}{\lambda_-(x)} - \frac{1}{2} \frac{\Delta_-^2(x)}{\lambda_+(x)}, \quad (3.1)$$

where

$$\|\dot{x}\|^2 = \int_0^1 \langle \dot{x}, \dot{x} \rangle_R ds, \quad \lambda_{\pm}(x) = \frac{a(x) - c(x) \pm \sqrt{(a(x) + c(x))^2 + 4b(x)^2}}{2}$$

and $\Delta_{\pm}(x)$ are suitable real maps depending also on Δ_y, Δ_t (see (2.2) and [2, p. 784]). Since $\mathcal{L}(x) = -\lambda_-(x)\lambda_+(x)$, necessarily $\lambda_+(x) > 0 > \lambda_-(x)$ for all $x \in \Omega^1(x_p, x_q)$, and thus

$$\mathcal{J}(x) \geq \frac{1}{2}\|\dot{x}\|^2 - \frac{1}{2} \frac{\Delta_-^2(x)}{\lambda_+(x)}.$$

Note also that, by the definition of $m(x)$ in (h_3) , we get

$$\lambda_+(x) \geq \frac{a(x) - c(x) + |a(x) + c(x)|}{2} = m(x) > 0.$$

Hence, (h_3) implies

$$\mathcal{J}(x) \geq \frac{1}{2}\|\dot{x}\|^2 - \frac{\Delta_-^2(x)}{2m(x)} \geq \frac{1}{2}\|\dot{x}\|^2 - \frac{\Delta_-^2(x)}{2} (h(x))^{-1} \quad \text{for all } x \in \Omega^1(x_p, x_q).$$

So, from [3, Theorem 1.1], it follows that \mathcal{J} is bounded from below and coercive (cf. (2.1) and (2.5)). Furthermore, by (h_2) and Lemma 2.4(b), the functional \mathcal{J} satisfies the Palais–Smale condition. Hence, Theorem 2.2 can be applied, and a geodesic connecting z_p with z_q is obtained. As z_p, z_q are arbitrary, the thesis follows. \square

An immediate consequence of Theorem 3.1 is the following result concerning some standard stationary spacetimes (cf. Example 1.3(e_4)).

Corollary 3.2. *Let $(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2, \langle \cdot, \cdot \rangle_L)$ be a standard stationary spacetime with $\langle \cdot, \cdot \rangle_L = \langle \cdot, \cdot \rangle_R + dy^2 + 2\delta(x)dydt - \beta(x)dt^2$, where $\delta, \beta : \mathcal{M}_0 \rightarrow \mathbb{R}$, $\beta(x) > 0$ in \mathcal{M}_0 . Assume also that*

- (s₁) $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$ is a complete Riemannian manifold;
- (s₂) there exist $\lambda_1, \lambda_2 \geq 0$, $k_1, k_2 \in \mathbb{R}$ and a point $x_0 \in \mathcal{M}_0$ such that

$$\beta(x) \leq \lambda_1 d^2(x, x_0) + k_1, \quad \delta(x) \leq \lambda_2 d(x, x_0) + k_2 \quad \text{for all } x \in \mathcal{M}_0.$$

Then, $(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2, \langle \cdot, \cdot \rangle_L)$ is geodesically connected.

Proof. As the standard stationary spacetime $(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2, \langle \cdot, \cdot \rangle_L)$ is a (GTS) with $A(x) \equiv 1$, $B(x) = \delta(x)$ and $C(x) = \beta(x)$, the thesis follows from Theorem 3.1. \square

Notice that Corollary 3.2 is a particular case of [1, Theorem 1.2] for general standard stationary manifolds $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ with $\langle \cdot, \cdot \rangle_L$ as in (1.2), which proof is based on fine estimates involving the metric coefficients. The following example shows the accurate character of this result.

Example 3.3. Let us consider \mathbb{R}^3 endowed with the following family of metrics:

$$\langle \cdot, \cdot \rangle_{L,\epsilon} = dx^2 + dy^2 - \beta_{\epsilon}(x)dt^2, \quad \epsilon \geq 0,$$

where $(x, y, t) \in \mathbb{R}^3$ and β_{ϵ} is a (positive) smooth function on \mathbb{R} such that

$$\begin{cases} \beta_{\epsilon}(x) = 1 + |x|^{2+\epsilon} & \text{if } x \in \mathbb{R} \setminus (-1, 1) \\ \beta_{\epsilon}([-1, 1]) \subset [1, 2]. \end{cases}$$

By Corollary 3.2 (with $\delta \equiv 0$), the spacetime is geodesically connected if $\epsilon = 0$. However, the spacetime is geodesically disconnected for any (and thus, for arbitrarily close to zero) strictly positive ϵ (see [3, Section 7]).

In order to give a more precise idea of the known results on geodesic connectedness in (GTS) by applying variational tools, let us review the corresponding results in [2]. In [2, Theorem 4.3], by using the expression (3.1) of \mathcal{J} , the geodesic connectedness of (GTS) is proven under assumptions (h_1) and (h_2) in Theorem 3.1, in addition to the following one:

(h'_3) $A(x) - C(x) > 0$ for all $x \in \mathcal{M}_0$ and the (positive) map $\frac{H(x)}{A(x) - C(x)}$ is at most quadratic.

Indeed, these conditions imply that \mathcal{J} is bounded from below and coercive, which allows us to apply Theorem 2.2 in view of Lemma 2.4(b).

As an immediate application of this result to Kerr–Schild spacetime (Example 1.3(e₃)), observe that here $A(x) - C(x) = 2V(x)$, $H(x) \equiv 1$ and $\mathcal{L}(x) \neq 0$ on $H^1(I, \mathcal{M}_0)$. Thus, the geodesic connectedness is ensured if V is strictly positive and $(2V(x))^{-1}$ is at most quadratic.

On the other hand, in [2, Theorem 4.4] we consider the simpler case, where $\mathcal{L}(x) \leq -\nu < 0$ for all $x \in H^1(I, \mathcal{M}_0)$ and $A(x) - C(x) < 0$ for all $x \in \mathcal{M}_0$.

Finally, notice that in [2] the growth assumption involves only the metric coefficients, and not the integrals in (2.2). This contrasts with [8, 9], where, in order to get the coercivity of \mathcal{J} , it is required that

$$\left| \frac{a(x)}{\mathcal{L}(x)} \right|, \quad \left| \frac{b(x)}{\mathcal{L}(x)} \right|, \quad \left| \frac{c(x)}{\mathcal{L}(x)} \right| \quad \text{are uniformly bounded on } H^1(I, \mathcal{M}_0).$$

Remark 3.4. Regarding to the case $A \equiv C$ left over in [2], if A (hence C) is always different from zero, then we are in the stationary case (Example 1.3(e₄)) with $\beta(x) = |A(x)|$.

In general, if $B \equiv 0$ and $H(x) = A(x)C(x) > 0$ with $A(x) > 0$ and $\beta(x) = C(x)$, then we have Example 1.3(e₄) in the static case. So, $\mathcal{J}(x) \geq J(x)$ on each $\Omega^1(x_p, x_q)$ and the optimal result in [3, Theorem 1.1] can be used. Let us point out that a direct use of (h'_3) for the particular case $A \equiv 1$ would give the desired result only for $\beta(x) < 1$.

If $A \equiv C \equiv 0$ then $\mathcal{L}(x) = b^2(x)$ and (GTS) becomes the more general type of warped product spacetimes, with fiber the two dimensional Lorentz–Minkowski spacetime \mathbb{L}^2 (see also [8, 11] and references therein). In this case we deal again with a functional as in (2.1), and we get global geodesic connectedness for the class of metrics $\langle \cdot, \cdot \rangle_R - 2\delta(x) dy dt$, where δ is a positive function with at most a quadratic growth (compare with [8, Appendix B]).

Moreover, if $a \equiv c$ on $H^1(I, \mathcal{M}_0)$, then

$$\mathcal{J}(x) \geq \frac{1}{2} \|\dot{x}\|^2 - \frac{\Delta_-^2}{|a(x)|}.$$

Hence, if $A(x) > 0$ in \mathcal{M}_0 , we obtain geodesic connectedness by assuming that $H(x)/A(x)$ grows at most quadratically in \mathcal{M}_0 (cf. (2.1) and (2.5)).

Remark 3.5. In [21] Piccione and Tausk generalize the Morse index theorem to semi–Riemannian manifolds admitting a smooth distribution spanned by commuting Killing vectors fields and containing a maximal negative distribution for the

given metric. In particular, they obtain Morse relations for standard stationary semi-Riemannian manifolds and, when the nondegeneracy condition $|\mathcal{L}(x)| > 0$ holds, for (GTS) (cf. [21, Theorems 4.6 and 4.8]). Clearly, Morse relations can be obtained also under our assumptions.

4. GEODESIC COMPLETENESS

In this section we establish and prove a result on geodesic completeness for (GTS) .

Theorem 4.1. *Let $(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2, \langle \cdot, \cdot \rangle_L)$ be a Gödel type spacetime such that:*

- (c₁) *$(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$ is a complete Riemannian manifold;*
- (c₂) *there exist $\lambda \geq 0$, $k \in \mathbb{R}$ and a point $x_0 \in \mathcal{M}_0$ such that the (positive) map*

$$\mu : x \in \mathcal{M}_0 \mapsto C(x) - A(x) + \sqrt{(A(x) + C(x))^2 + 4B(x)^2} \in \mathbb{R}$$

satisfies

$$1/\mu(x) \leq \lambda d^2(x, x_0) + k \quad \text{for all } x \in \mathcal{M}_0. \quad (4.1)$$

Then, $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ is geodesically complete.

Proof. Let $z : [0, T) \rightarrow \mathcal{M}$, $z(s) = (x(s), y(s), t(s))$, be an inextendible geodesic. Arguing by contradiction, it is enough to prove that if $T < +\infty$ then the $\langle \cdot, \cdot \rangle_R$ -length of $x(s)$ is bounded, and so, z can be extended to T against the maximality assumption (see [20, Lemma 5.8]).

As ∂_y and ∂_t are Killing vector fields, there exist constants $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{cases} A(x)\dot{y} + B(x)\dot{t} \equiv c_1 \\ B(x)\dot{y} - C(x)\dot{t} \equiv c_2 \end{cases} \quad \text{for all } s \in [0, T), \quad (4.2)$$

with

$$\mathcal{S}(x) = \begin{pmatrix} A(x) & B(x) \\ B(x) & -C(x) \end{pmatrix} \quad (4.3)$$

symmetric matrix with $\det \mathcal{S}(x) = -H(x) < 0$.

Furthermore, as z is a geodesic, there exists a constant $E_z \in \mathbb{R}$ such that

$$\langle \dot{z}, \dot{z} \rangle_L = \langle \dot{x}, \dot{x} \rangle_R + A(x)\dot{y}^2 + 2B(x)\dot{y}\dot{t} - C(x)\dot{t}^2 \equiv E_z \quad \text{for all } s \in [0, T). \quad (4.4)$$

Thus, by (4.2) and (4.4) we get

$$\langle \dot{x}, \dot{x} \rangle_R + c_1\dot{y} + c_2\dot{t} = E_z \quad \text{for all } s \in [0, T). \quad (4.5)$$

On the other hand, by (4.2) and (1.4) we have

$$\dot{y} = \frac{c_1 C(x) + c_2 B(x)}{H(x)}, \quad \dot{t} = \frac{c_1 B(x) - c_2 A(x)}{H(x)}.$$

Whence, by (4.5) and using the notation $\|\dot{x}\|_R^2 := \langle \dot{x}, \dot{x} \rangle_R$, we get

$$\|\dot{x}\|_R^2 = E_z + \frac{c_2^2 A(x) - c_1^2 C(x) - 2c_1 c_2 B(x)}{H(x)}. \quad (4.6)$$

Note that the symmetric matrix $\mathcal{S}(x)$ in (4.3) admits two (non-null) real eigenvalues

$$\Lambda_{\pm}(x) = \frac{A(x) - C(x) \pm \sqrt{(A(x) + C(x))^2 + 4B^2(x)}}{2}, \quad \text{with } \Lambda_+(x) > 0 > \Lambda_-(x).$$

Recall that by standard arguments there exists an orthogonal matrix $Q(x)$ such that

$$Q(x)^T \begin{pmatrix} A(x) & B(x) \\ B(x) & -C(x) \end{pmatrix} Q(x) = \begin{pmatrix} \Lambda_+(x) & 0 \\ 0 & \Lambda_-(x) \end{pmatrix}.$$

Let us denote $(\tilde{c}_1, \tilde{c}_2) = (c_1^2 + c_2^2)^{-1/2}(c_1, c_2)$ and $(\tilde{c}_1(x), \tilde{c}_2(x)) = (\tilde{c}_1, \tilde{c}_2)Q(x)$. By definition we have $\mu(x) = -2\Lambda_-(x)$, and, by the orthogonality of $Q(x)$, we have $[\tilde{c}_i(x)]^2 \leq 1$ for $i \in \{1, 2\}$. So, we can rewrite (4.6) as:

$$\begin{aligned} \|\dot{x}\|_R^2 &= E_z + \frac{\begin{pmatrix} c_1 & c_2 \end{pmatrix} \begin{pmatrix} A(x) & B(x) \\ B(x) & -C(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}{H(x)} \\ &= E_z + \frac{\begin{pmatrix} c_1 & c_2 \end{pmatrix} Q(x) \begin{pmatrix} \Lambda_+(x) & 0 \\ 0 & \Lambda_-(x) \end{pmatrix} Q(x)^T \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}{H(x)} \\ &= E_z + (c_1^2 + c_2^2) \frac{\begin{pmatrix} \tilde{c}_1 & \tilde{c}_2 \end{pmatrix} Q(x) \begin{pmatrix} \Lambda_+(x) & 0 \\ 0 & \Lambda_-(x) \end{pmatrix} Q(x)^T \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix}}{H(x)} \\ &= E_z + (c_1^2 + c_2^2) \frac{\begin{pmatrix} \tilde{c}_1(x) & \tilde{c}_2(x) \end{pmatrix} \begin{pmatrix} \Lambda_+(x) & 0 \\ 0 & \Lambda_-(x) \end{pmatrix} \begin{pmatrix} \tilde{c}_1(x) \\ \tilde{c}_2(x) \end{pmatrix}}{-\Lambda_+(x)\Lambda_-(x)} \\ &= E_z - (c_1^2 + c_2^2) \left(\frac{[\tilde{c}_1(x)]^2}{\Lambda_-(x)} + \frac{[\tilde{c}_2(x)]^2}{\Lambda_+(x)} \right) \\ &\leq E_z - \frac{c_1^2 + c_2^2}{\Lambda_-(x)} = E_z + 2 \frac{c_1^2 + c_2^2}{\mu(x)}. \end{aligned}$$

Thus, by (4.1) there exist suitable constants $\bar{\lambda}, \bar{k} > 0$ such that:

$$\|\dot{x}(s)\|_R \leq \bar{\lambda}d(x(s), x(0)) + \bar{k} \leq \bar{\lambda} \int_0^s \|\dot{x}(r)\|_R dr + \bar{k} \quad \text{for all } s \in [0, T].$$

In conclusion, we obtain

$$\log \left(\bar{\lambda} \int_0^s \|\dot{x}(r)\|_R dr + \bar{k} \right) - \log(\bar{k}) \leq \bar{\lambda}s \leq \bar{\lambda}T \quad \text{for all } s \in [0, T]$$

which implies the boundedness of the $\langle \cdot, \cdot \rangle_R$ -length of $x(s)$ in $[0, T]$, as required. \square

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